NYU-TH-99/09/03

Singleton field theory and Flato - Fronsdal dipole equation

Andrei Starinets[1]

Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA

Abstract

We study solutions of the equations $(\triangle - \lambda)\phi = 0$ and $(\triangle - \lambda)^2\phi = 0$ in global coordinates on the covering space $CAdS_d$ of the d-dimensional Anti de-Sitter space subject to various boundary conditions and their connection to the unitary irreducible representations of $\widetilde{SO}(d-1,2)$. The "vanishing flux" boundary conditions at spatial infinity lead to the standard quantization scheme for $CAdS_d$ in which solutions of the second- and the fourth-order equations are equivalent. To include fields realizing the singleton unitary representation in the bulk of $CAdS_d$ one has to relax the boundary conditions thus allowing for the nontrivial space of solutions of the dipole equation known as the Gupta - Bleuler triplet. We obtain explicit expressions for the modes of the Gupta - Bleuler triplet and the corresponding two-point function. To avoid negative-energy states one must also introduce an additional constraint in the space of solutions of the dipole equation.

[1]e-mail: andrei.starinets@physics.nyu.edu

1 Introduction

The purpose of this paper is to obtain explicit solutions in global coordinates to the second- and fourth-order field equations on the d-dimensional covering of the Anti de Sitter space, and to establish their connection to the unitary irreducible representations of the isometry group $\widetilde{SO}(d-1,2)$. One may be interested in finding explicit expressions for the modes of the singleton field theory in a view of AdS/CFT correspondence [1], its Lorentzian version [2] and other related issues [3], [4], [5]. Anti de Sitter space $AdS_d = SO_0(d-1,2)/SO_0(d-1,1)$ possesses two well-known properties — rather pathological ones from the physical point of view: it has closed timelike curves and lacks a global Cauchy surface [6]. The first pathology can be cured by considering covering space $CAdS_d$; the second one requires introduction of the special boundary conditions for the solutions of field equations at spatial infinity [7], [8]. We are interested therefore in studying $CAdS_d$ harmonics classified according to the representations of the covering group $\widetilde{SO}(d-1,2)$ of $SO(d-1,2)$.

Representation theory In the literature[2], one can find numerous examples of classification of UIRs of Lie algebra $so(d-1,2)$ (and corresponding superalgebra), especially for $d=4,5$ [10](review and references can be found in [11]). UIRs of $\widetilde{SO}(d-1,2)$ can be decomposed into the direct sum of UIRs of its maximal compact subgroup, $SO(d-1)\times SO(2)$ and are uniquely characterized by the eigenvalue ω of the $SO(2)$ generator and by weights of $SO(d-1)$. The eigenvalue ω is always[3] of the form $\omega = E_0 + k$, where k is a nonnegative integer. The condition of unitarity imposes constraint on values of E_0,

$$E_0 \geq E_0^{min}, \tag{1}$$

where E_0^{min} depends on the dimension of $CAdS_d$ and the type of representation. UIRs of $\widetilde{SO}(d-1,2)$ exhibit certain unusual (by the standards of the UIRs of compact groups) properties. When the unitarity bound (1) is saturated, the number of components in the multiplet is dramatically reduced in comparison with the multiplet characterized by any $E_0 > E_0^{min}$. More precisely, if we denote states in a given multiplet by $|E_0, l, k>$, where l is a (set of) quantum numbers of $SO(d-1)$ and $k = 0, 1, \ldots$, then for $E_0 = E_0^{min}$ all states $|E_0^{min}, l, k>$ with $k>0$ have zero norm. This peculiar UIR (first discussed in a group-theoretical approach by Ehrman [12] and Dirac [13] for $d=4$) is called the *singleton representation*[4]. The tensor product of two singleton representations decomposes into an infinite set of massless UIRs [15]; hence singletons

[2] Mathematical literature provides comprehensive treatment of harmonic analysis on AdS_d (summary and references can be found in [9] which differs from the physically interesting case of field theory on $CAdS_d$.

[3] Considering only physically interesting UIRs, i.e. those with real positive ω.

[4] Under this name it appeared in Ehrman's paper [12]. In dimensions $d=5$ and $d=7$ the corresponding UIRs were named "doubletons"[14].

may be regarded as true fundamental degrees of freedom in the appropriate field theory. This observation served as a powerful motivation for the subsequent development of the singleton field theory in the bulk of $CAdS_4$ [4]. Also, singleton short multiplets (or supermultiplets, if one considers supersymmetric extensions of $so(d-1,2)$) received much attention in the '80s in connection with the dimensional reduction of supergravity theories [14], [16], [17]. Singleton representation in the light of AdS/CFT correspondence is considered in [18, 19].

Field theory on $CAdS_d$ Field theory on four-dimensional[5] Anti de-Sitter space was extensively investigated some time ago by various authors [7], [8],[20] —[23]. Imposing vanishing flux boundary conditions (thoroughly described in [8]) at the $CAdS_d$ spatial infinity, one finds that for $E_0 > E_0^{min}$ solutions of the wave equation on $CAdS_d$ are in one-to-one correspondence with the UIRs of $\widetilde{SO}(d-1,2)$ just like ordinary spherical harmonics on $S^d = SO(d+1)/SO(d)$ are associated with UIRs of $SO(d)$. More precisely, singular points of the differential equation $\left(\triangle_{CAdS_d} - \lambda(E_0)\right)\phi = 0$ in the interval

$$\frac{d-3}{2} < E_0 < \frac{d+1}{2} \tag{2}$$

are of the limit-circle [24] type. Vanishing flux boundary conditions are chosen in such a way that Cauchy problem on $CAdS_d$ becomes well-defined [7, 8]. This ultimately gives frequency quantization and establishes connection with $\widetilde{SO}(d-1,2)$ UIRs. Two sets of square-integrable $CAdS_d$ harmonics exist for E_0 in the interval (2). Outside the interval (2) singular points are of the limit-point type and the condition of square-integrability leaves only one set of modes for $E_0 = (d-3)/2$ and $E_0 \geq (d+1)/2$. The theory is then quantized following the standard scheme [25].

Singleton field theory The modes at $E_0 = (d-3)/2$ allowed by the traditional (vanishing flux and square-integrability) boundary conditions transform according to the $D(\frac{d+1}{2}, 0)$ UIR of $\widetilde{SO}(d-1,2)$. Those boundary conditions therefore do not allow realization of the singleton UIR in the bulk of $CAdS_d$.

However, the second-order equation at $E_0 = (d-3)/2$ also admits another, non square-integrable, solution which has logarithmic singularity at the boundary of $CAdS_d$. The logarithmic singularity disappears and the solution becomes a polynomial if we impose frequency quantization $\omega = (d-3)/2 + l$. Thus we obtain non square-integrable set[6] of the *singleton modes* transforming according to $D(\frac{d-3}{2}, 0)$. The total space of solutions at $E_0 = (d-3)/2$ corresponds

[5]Generalization to arbitrary d is straightforward. For explicit solutions of the second-order equation, see [2] and Section 2 of the present paper.

[6] Non square-integrability here does not signal the appearance of the continuous spectrum as can be seen clearly by recasting the eigenvalue equation into the Schrödinger form (see Appendix (C)).

to the indecomposable representation $D(\frac{d-3}{2},0) \to D(\frac{d+1}{2},0)$. Singleton modes fall off more slowly at spatial infinity than any solution with $E > E_0^{min}$. In this sense they decouple from the rapidly decreasing "gauge modes" (solutions bearing $D(\frac{d+1}{2},0)$ labels) only at the boundary of $CAdS_d$ [7].

"Singleton modes" on $CAdS_4$ were introduced and studied by C.Fronsdal, M.Flato and collaborators in the series of papers [27] — [32] (see also [4] and references therein). Studying the corresponding two-point function, Flato and Fronsdal [30] proposed a fourth-order wave equation to describe a field theory of "singletons" (which was regarded as a gauge theory in the bulk of AdS_4) and formulated the appropriate Lagrangian formalism. Quantization of the theory including the BRST approach was also developed. Flato - Fronsdal wave equation is supposed to provide the Gupta - Bleuler triplet of "scalar", "singleton" and "gauge" modes [30], [33], [4] required for quantization of the theory. The second-order equation then serves as an analog of the Lorentz gauge condition. Explicit solutions of the fourth-order equation carrying $\widetilde{SO}(d-1,2)$ UIRs have not been previously obtained.

In this paper, we study solutions of the second- and the fourth-order free field equations on $CAdS_d$ in global coordinates. In Section 2 we consider modes of the second-order equation subject to vanishing flux boundary conditions, and the singleton modes. We compute the corresponding two-point function — first by solving the equation directly and then by using the mode expansion — and discuss the $E_0 \to (d-3)/2$ limit. Solutions of the Flato - Fronsdal dipole equation are obtained in Section 3, modes of the Gupta - Bleuler triplet are identified. We show that the constraint must be imposed on the space of solutions of the fourth-order equation to eliminate the "scalar sector" solutions with the lowest frequency[8]. We also obtain the singleton two-point function and its decomposition into the sum of Gupta - Bleuler triplet modes.

2 $CAdS_d$ harmonics and boundary conditions

In R^{d+1} with metric $\eta_{ij} = (+,-,...,-,+)$ d-dimensional Anti-de Sitter space AdS_d can be realized as hyperboloid

$$X_0^2 + X_d^2 - \vec{X}^2 = 1/a^2, \tag{3}$$

[7] The situation described above can be compared to other examples provided by harmonic analysis on noncompact groups where certain members of the *discrete* series can not be realized as elements of the Hilbert space of square-integrable functions on the corresponding homogeneous space and thus do not appear in the Plancherel formula for the group. Comprehensive analysis for the case of $SO(2,1)$ can be found in Bargmann's paper [26]. The singleton representation of $\widetilde{SO}(d-1,2)$ is an example of such situation.

[8] Those solutions are not polynomial. The action of energy-lowering operators on them leads to the states with negative energy.

where $a = const$ is the "inverse radius" of AdS_d. It is a space of constant curvature $R = d(d-1)a^2$ which is locally characterized by

$$R_{ijkl} = \frac{R}{12}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right), \tag{4}$$

The Ricci tensor is proportional to the metric,

$$R_{ij} = \frac{R}{d}g_{ij} = (d-1)a^2 g_{ij}. \tag{5}$$

AdS_d can be viewed as a solution of Einstein's equations

$$R_{ij} - \frac{1}{2}g_{ij}R = -\Lambda g_{ij}, \tag{6}$$

with

$$\Lambda = \frac{d-2}{2d}R = \frac{(d-2)(d-1)}{2}a^2.$$

It is convenient to use the parametrization

$$X^0 = \frac{\sec r}{a}\sin t, \tag{7}$$

$$X^d = -\frac{\sec r}{a}\cos t, \tag{8}$$

$$X^i = \frac{\tan r}{ar}z^i, \tag{9}$$

where $i = 1, \ldots, d-1$, $r^2 = \vec{z}\vec{z}$. We shall use $d-1$ —dimensional spherical coordinates for z^i. The range of t is $[-\pi, \pi)$ for AdS_d and $(-\infty, \infty)$ for its covering space, $CAdS_d$, the range of r is $[0, \pi/2)$, where $r = \pi/2$ corresponds to spatial infinity. The metric can be written as

$$ds^2 = \frac{\sec^2 r}{a^2}\left(dt^2 - dr^2 - \sin^2 r d\Omega^2_{d-2}\right). \tag{10}$$

The metric on slices $t = const$ is conformally equivalent to a half of the sphere S^{d-1} with scalar curvature $R = -\frac{d-2}{d}R_{AdS_d}$:

$$ds^2 \mid_{t=const} = -\frac{\sec^2 r}{a^2}d\Omega^2_{d-1}. \tag{11}$$

This fact is helpful in discussion of the boundary conditions on $CAdS_d$.

Generators of the $CAdS_d$ isometry group,

$$L_{AB} = X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A}, \tag{12}$$

$A, B = 0, ...d$, can be written in intrinsic coordinates (7) — (9) as

$$L_{0d} = \frac{\partial}{\partial t}, \tag{13}$$

$$L_{i0} = -\cos t \sin r \frac{z^i}{r}\frac{\partial}{\partial t} - \sin t \left[\cos r \frac{z^i z^k}{r^2}\frac{\partial}{\partial z^k} + \frac{r}{\sin r}\left(\frac{\partial}{\partial z^i} - \frac{z^i z^k}{r^2}\frac{\partial}{\partial z^k}\right)\right], \tag{14}$$

$$L_{id} = -\sin t \sin r \frac{z^i}{r}\frac{\partial}{\partial t} + \cos t \left[\cos r \frac{z^i z^k}{r^2}\frac{\partial}{\partial z^k} + \frac{r}{\sin r}\left(\frac{\partial}{\partial z^i} - \frac{z^i z^k}{r^2}\frac{\partial}{\partial z^k}\right)\right], \tag{15}$$

$$L_{ij} = z^j \frac{\partial}{\partial z^i} - z^i \frac{\partial}{\partial z^j}. \tag{16}$$

Elements of the corresponding $so(d-1,2)$ algebra, $M_{AB} = iL_{AB}$, satisfy

$$[M_{AB}, M_{CD}] = i\left(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}\right). \tag{17}$$

The operators M_{ik} and M_{0d}, $i,k = 1,...,d-1$, generate subalgebra $so(d-1)\otimes u(1)$ of $so(d-1,2)$, M_{0d} being identified as the energy operator on $CAdS_d$. Representations can be built by acting by the energy raising (lowering) operators,

$$M_k^{\pm} = iM_{0k} \mp M_{kd}, \tag{18}$$

on the lowest energy state satisfying $M_i^-|E_0, j_1, \dots j_{[\frac{d-1}{2}]} >= 0$. Explicit expressions for M_{ik} in the $d=4$ case (in coordinates r,t,θ,ϕ) are

$$M_k^{\pm} = -e^{\mp it}\sin r \frac{z^i}{r}\partial_t \mp ie^{\mp it}R_i, \tag{19}$$

where

$$R_1 = \sin\theta\cos\varphi\cos r\partial_r + \frac{\cos\theta\cos\varphi}{\sin r}\partial_\theta - \frac{\sin\varphi}{\sin r\sin\theta}\partial_\varphi, \tag{20}$$

$$R_2 = \sin\theta\sin\varphi\cos r\partial_r + \frac{\cos\theta\sin\varphi}{\sin r}\partial_\theta + \frac{\cos\varphi}{\sin r\sin\theta}\partial_\varphi, \tag{21}$$

$$R_3 = \cos\theta\cos r\partial_r - \frac{\sin\theta}{\sin r}\partial_\theta \tag{22}$$

We consider equation of the form

$$(\Box_{CAdS_d} - \lambda)\,\phi = 0, \tag{23}$$

where $\lambda = m_0^2$ is a mass parameter. Acting on functions, $\Box_{CAdS_d}$ gives

$$\Box f = -g^{ij}\nabla_j\nabla_i f = -g^{ij}\partial_i\partial_j f + g^{ij}\Gamma^l_{ij}\partial_l f = -\frac{1}{|g|^{1/2}}\partial_i\left(g^{ij}|g|^{1/2}\partial_j f\right) \tag{24}$$

In coordinates (7) — (9) the Laplacian on $CAdS_d$ is

$$\Box_{CAdS_d} = -a^2\cos^2 r\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right) + (d-2)a^2\cot r\frac{\partial}{\partial r} + a^2\cot^2 r\triangle_{S^{d-2}}, \tag{25}$$

where $\triangle_{S^{d-2}}$ is the Laplacian on the unit sphere S^{d-2}.

A plane-wave solution of

$$\Box F \;=\; \lambda F \tag{26}$$

can be written as $F(r,t,\vec{\theta}) = e^{-i\omega t} f(r) Y(\vec{\theta})$, where $Y(\vec{\theta})$ is an eigenfunction of $\triangle_{S^{d-2}}$,

$$\triangle_{S^{d-2}} \mathbf{Y}(\theta) \;=\; \Lambda \mathbf{Y}(\theta), \tag{27}$$

where $\Lambda = -l(l+d-3)$, l being the highest weight of irreducible representation of $SO(d-1)$.

The equation for $f(r)$ then reads[9]

$$Qf \;=\; f''(r) + \frac{2(d-2)}{\sin 2r} f'(r) - \left[\frac{l(l+d-3)}{\sin^2 r} + \frac{\lambda_p}{a^2 \cos^2 r} - \omega^2 \right] f(r) = 0. \tag{28}$$

The singular points are $r = 0$ and $r = \pi/2$. The roots of the indicial equation at $r = 0$ and $r = \pi/2$ are correspondingly $\alpha_1^0 = l$, $\alpha_2^0 = 3 - d - l$ and

$$\alpha_{1,2}^{\pi/2} = \frac{d-1}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 + \frac{\lambda}{a^2}}. \tag{29}$$

For a scalar field it is convenient[10] to introduce the parameter E_0 such that[11]

$$m_0^2 = a^2 E_0 \left(E_0 - d + 1\right). \tag{30}$$

Then in terms of E_0 roots of the indicial equation are

$$\alpha_1^{\pi/2} \;=\; E_0, \tag{31}$$

$$\alpha_2^{\pi/2} \;=\; d - 1 - E_0. \tag{32}$$

We are looking for solutions of (26) on $CAdS_d$ square-integrable with respect to the metric

$$(F_1, F_2) \;=\; i \int d^{d-1}x \sqrt{-g} g^{0\nu} \left(\bar{F}_1 \partial_\nu F_2 - F_2 \partial_\nu \bar{F}_1 \right) \tag{33}$$

In coordinates (7) — (9) this becomes

$$(F_1, F_2) \;=\; i \int_0^{\pi/2} \left(\frac{\tan r}{a} \right)^{d-2} dr \int d\Omega_{d-2} \left(\bar{F}_1 \partial_t F_2 - F_2 \partial_t \bar{F}_1 \right). \tag{34}$$

One can see that for $r = 0$ the solution with index α_1^0 is nonsingular and square-integrable for $l > (1-d)/2$ (i.e. for all relevant l), the solution with α_2^0 is nonsingular for $l \le 3 - d$ and

[9] It can also be written in the form of the Schrödinger equation. In Appendix (C) we discuss interpretation of the “singleton modes” in this picture.

[10] Acting on p-forms, the second-order Casimir operator of $so(d-1,2)$ reads $C_2 \varphi_p = (E_0 - p)(E_0 - d + 1 + p)\varphi_p = m_p^2 / a^2 \varphi_p$ [18].

[11] For the scalar field it is sometimes convenient to separate the conformal coupling term from m_0^2 and write $m_0^2 = \frac{d-2}{4(d-1)} R + m_c^2$. Then $m_c^2 \;=\; \lambda_c = a^2 \left(E_0 - \frac{d}{2}\right)\left(E_0 - \frac{d}{2} + 1\right)$. This dependence is sketched in Figure (1).

square-integrable for $l < (5-d)/2$. Since we need nonsingular square-integrable solutions for all nonnegative l, we choose[12] solution with α_1^0.

For $r = \pi/2$ solutions with index $\alpha_1^{\pi/2}$ are nonsingular if $E_0 \geq 0$ and square-integrable if $E_0 > (d-3)/2$. Solutions with index $\alpha_2^{\pi/2}$ are nonsingular if $E_0 \geq d-1$ and square-integrable if $E_0 > (d+1)/2$. One can easily find solutions of (28) with the desired asymptotics. Writing $f(r)$ as $f(r) = \sin^l r \cos^{E_0} r g(r)$, one reduces (28) to a hypergeometric equation in $x = \sin^2 r$:

$$x(x-1)g''(x) + g'\left(x(l+E_0+1) - l - \frac{d-1}{2}\right) + \frac{(l+E_0)^2 - \omega^2}{4} g(x) \;=\; 0. \tag{35}$$

The solution with correct asymptotics at $r = 0$ is[13]

$$f(r) \;=\; \sin^l r \cos^{E_0} r g(r) \;=\; \sin^l r \cos^{E_0} r \, {}_2F_1\left(\frac{l+E_0+\omega}{2}, \frac{l+E_0-\omega}{2}; l + \frac{d-1}{2}; \sin^2 r\right) \tag{36}$$

Consider now the behavior of (36) at $r \to \pi/2$. Using a transformation property (108) of hypergeometric functions we can conveniently express $f(r)$ as

$$\begin{aligned} f(r) \;&=\; C_1 \sin^l r \cos^{E_0} r {}_2F_1\left(\frac{l+E_0+\omega}{2}, \frac{l+E_0-\omega}{2}; E_0 - \frac{d-3}{2}; \cos^2 r\right) \\ &+\; C_2 \sin^l r \cos^{d-1-E_0} r \, {}_2F_1\left(\frac{l-E_0-\omega+d-1}{2}, \frac{l+E_0+\omega+d-1}{2}; \frac{d+1}{2} - E_0; \cos^2 r\right), \end{aligned}$$

where

$$C_1 \;=\; \frac{\Gamma\left(l+\frac{d-1}{2}\right)\Gamma\left(\frac{d-1}{2} - E_0\right)}{\Gamma\left(\frac{l+d-1-E_0-\omega}{2}\right)\Gamma\left(\frac{l+d-1-E_0+\omega}{2}\right)}, \quad C_2 \;=\; \frac{\Gamma\left(l+\frac{d-1}{2}\right)\Gamma\left(E_0 - \frac{d-1}{2}\right)}{\Gamma\left(\frac{l+E_0+\omega}{2}\right)\Gamma\left(\frac{l+E_0-\omega}{2}\right)}. \tag{37}$$

2.1 Solution in the interval $\frac{d-3}{2} < E_0 < \frac{d+1}{2}$

In the range

$$\frac{d-3}{2} < E_0 < \frac{d+1}{2} \tag{38}$$

(for $d > 1$) the singular point $r = \pi/2$ is of the limit-circle type (asymptotics with $\alpha_1^{\pi/2}$ and $\alpha_2^{\pi/2}$ are equally admissible) so we need to specify boundary conditions.

As mentioned in the Introduction (and discussed in detail in [8]), the Cauchy problem on $CAdS_d$ is ill-defined since time development of fields can be affected by the information crossing spatial infinity $r = \pi/2$ in finite time. To make evolution predictable, we have to impose the condition that the flux through the boundary $r = \pi/2$ vanishes. Calculation of the flux proceeds exactly as in the four-dimensional case [8] and one finds that the requirement of vanishing flux is equivalent to setting either C_1 or C_2 in (36) to zero.

[12] From now on we assume $d > 2$. Two-dimensional case is considered in Appendix (B).

[13] Due to the transformation property (107) one can also write this solution in the form $\sin^l r \cos^{d-1-E_0} r_2F_1\left(\frac{l-E_0-\omega+d-1}{2}, \frac{l-E_0+\omega+d-1}{2}; l + \frac{d-1}{2}; \sin^2 r\right)$.

Another way to describe these boundary conditions is to consider the conformal map (11) from $CAdS_d$ to the half of S^{d-1}. The boundary of $CAdS_d$ is mapped into the equator of S^{d-1}. Fields transform according to

$$\Phi = \cos^{1-d/2} r \; \Phi_{CAdS_d}.$$

Near the equator of S^{d-1} one has then

$$f(r) \sim C_1 \cos^{E_0+1-\frac{d}{2}} r + C_2 \cos^{\frac{d}{2}-E_0} r + \text{higher powers of } \cos r$$

At $E_0 = d/2 - 1$ (massless scalar field) this becomes simply

$$f(r) \sim C_1 + C_2 \cos r + \text{higher powers of } \cos r$$

Thus conditions $C_1 = 0$ or $C_2 = 0$ correspond to ordinary Dirichlet or Neumann boundary conditions, respectively.

We have therefore two sets of modes on $CAdS_d$ corresponding to "Dirichlet" ($C_1 = 0$) or "Neumann" ($C_2 = 0$) boundary conditions.

Dirichlet boundary condition Condition $C_1 = 0$ leads to quantization of ω: $\omega_k = d - 1 - E_0 + l + 2k$, where $k = 0, 1, \ldots$. The second coefficient becomes

$$C_2(\omega_k) = \Gamma\left(l + \frac{d-1}{2}\right)\Gamma\left(E_0 - \frac{d-1}{2}\right)/\Gamma\left(\frac{d-1}{2} + l + k\right)\Gamma\left(E_0 - \frac{d-1}{2} - k\right).$$

The solution is

$$\begin{aligned} f(r) &= \sin^l r \cos^{d-1-E_0} r\, {}_2F_1\left(-k, d-1-E_0+l+k; l+\frac{d-1}{2}; \sin^2 r\right) \\ &= \sin^l r \cos^{d-1-E_0} r \frac{k!}{\left(l+\frac{d-1}{2}\right)_k} P_k^{(l+\frac{d-3}{2}, \frac{d-1}{2}-E_0)}(\cos 2r), \end{aligned} \tag{39}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer's symbol. Thus, the first set of solutions[14] of equation (26) is

$$F^D_{E_0,l,k}(r,t,\theta) = C_{E_0,l,k} e^{-i\omega_k t} f^-_k(r) \mathbf{Y}(\theta), \tag{40}$$

where

$$f^D_{E_0,l,k}(r) = \sin^l r \cos^{d-1-E_0} r P_k^{(l+\frac{d-3}{2}, \frac{d-1}{2}-E_0)}(\cos 2r), \tag{41}$$

$$C^2_{E_0,l,k} = \frac{a^{d-2}\Gamma(d-1-E_0+l+k)\, k!}{\Gamma\left(\frac{d-1}{2}+l+k\right)\Gamma\left(\frac{d+1}{2}-E_0+k\right)}, \tag{42}$$

$$\omega_k = d - 1 - E_0 + l + 2k, \quad k = 0, 1, 2, \ldots. \tag{43}$$

[14]Normalized with respect to the metric (34).

Neumann boundary condition Condition $C_2 = 0$ gives frequency quantization $\omega_k = E_0 + l + 2k$, $k = 0, 1, \ldots$. Coefficient C_1 is

$$C_1(\omega_k) \;=\; \Gamma\left(l + \frac{d-1}{2}\right)\Gamma\left(\frac{d-1}{2} - E_0\right)/\Gamma\left(\frac{d-1}{2} + l + k\right)\Gamma\left(\frac{d-1}{2} - E_0 - k\right).$$

The second set of solutions is therefore

$$F^N_{E_0,l,k}(r,t,\theta) \;=\; C_{E_0,l,k} e^{-i\omega_k t} f^+_k(r)\mathbf{Y}(\theta), \tag{44}$$

where

$$f^N_{E_0,l,k}(r) \;=\; \sin^l r \cos^{E_0} r \; P_k^{(l+\frac{d-3}{2}, E_0-\frac{d-1}{2})}(\cos 2r), \tag{45}$$

$$C^2_{E_0,l,k} \;=\; \frac{a^{d-2}\Gamma\left(E_0 + l + k\right) k!}{\Gamma\left(\frac{d-1}{2} + l + k\right)\Gamma\left(E_0 - \frac{d-3}{2} + k\right)}, \tag{46}$$

$$\omega_k \;=\; E_0 + l + 2k, \quad k = 0, 1, 2, \ldots. \tag{47}$$

Massless modes Massless conformally coupled scalar modes correspond to $E_0 = d/2$ or $E_0 = d/2 - 1$ and are given by $F(r,t,\theta) = f_k(r) e^{-i\omega_k t}\mathbf{Y}(\theta)$, where for $E_0 = d/2$ $\omega_k = d/2 + l + 2k$,

$$f_k^{E_0=d/2}(r) \;=\; A \sin^l r \cos^{\frac{d}{2}-1} r C_{2k+1}^{l+\frac{d}{2}-1}(\cos r), \tag{48}$$

where C_k^α are Gegenbauer polynomials and

$$A^2 \;=\; a^{d-2}\Gamma(k+3/2)k!/\pi\,(l+d/2+1)_{k+1}\,(l+d/2-1)_{k+1/2}\,.$$

for $E_0 = d/2 - 1$ we have $\omega_k = d/2 - 1 + l + 2k$,

$$f_k^{E_0=d/2-1}(r) \;=\; B \sin^l r \cos^{\frac{d}{2}-1} r C_{2k}^{l+\frac{d}{2}-1}(\cos r), \tag{49}$$

where $B^2 \;=\; a^{d-2}\Gamma(k+1/2)k!/\pi\,(l+d/2+1)_{k+1}\,(l+d/2-1)_{k+1/2}\,.$

2.2 Solution outside the interval $\frac{d-3}{2} < E_0 < \frac{d+1}{2}$

Outside the interval $\frac{d-3}{2} < E_0 < \frac{d+1}{2}$ singular point $r = \pi/2$ of the differential equation (28) is of the limit-point type (boundary condition becomes the condition of square-integrability on $CAdS_d$). This means that for $E_0 < \frac{d-3}{2}$ ($E_0 > \frac{d+1}{2}$) coefficient C_1 (C_2) must be set to zero and, correspondingly, only the Dirichlet set (Neumann set) is acceptable. We note that the $E_0 < \frac{d-3}{2}$ Dirichlet set is equivalent to the $E_0 > \frac{d+1}{2}$ Neumann set and thus for $m_0^2/a^2 > m^2_{BF}/a^2 + 1 = -(d-1)^2/4 + 1$ there exists only one set of normalizable solutions on $CAdS_d$.

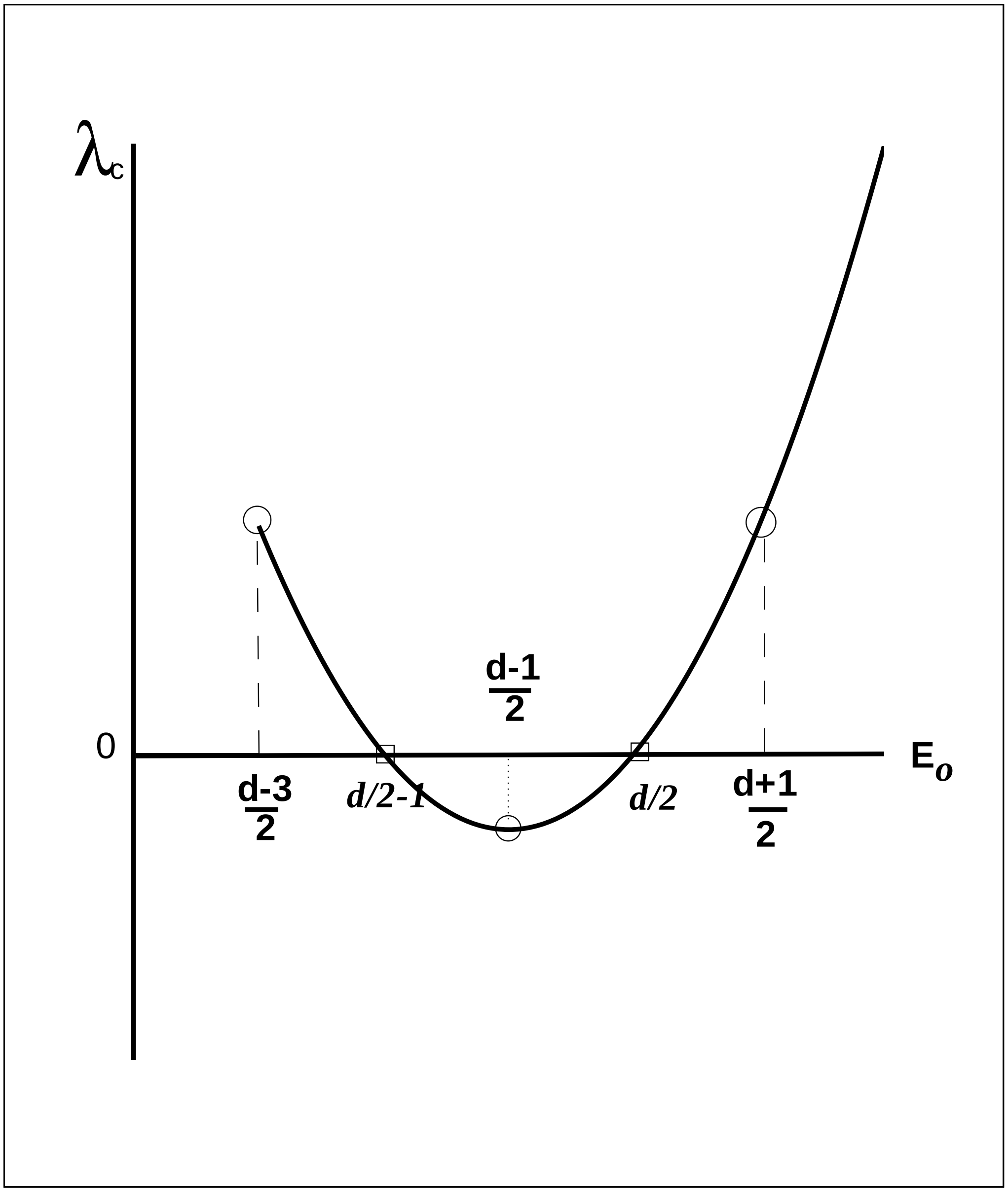

Figure 1: Mass squared $\lambda_c = m_c^2$ of the conformally coupled scalar as a function of E_0. The singleton UIR corresponds to $E_0 = \frac{d-3}{2}$. Two sets of $CAdS_d$ harmonics exist for $\frac{d-3}{2} < E_0 < \frac{d+1}{2}$. Modes with $d/2 - 1 < E_0 < d/2$ have negative mass squared, $m_{BF}^2 \leq m_c^2 < 0$, where $m_{BF}^2 = -a^2/4$ is the Breitenlohner - Freedman bound.

2.3 Solution for $E_0 = \frac{d-1}{2} + m, m \in Z$

When the difference between the roots (31) – (32) of the indicial equation becomes an integer number, i.e. when

$$E_0 = \frac{d-1}{2} + m, \quad m \in Z, \tag{50}$$

we may expect logarithmic terms in the solution of equation (28). For example, when $E_0 = (d-1)/2$, (36) degenerates to

$$\begin{aligned} f(r) &= \frac{\Gamma\left(l+\frac{d-1}{2}\right)}{\Gamma(a)\,\Gamma(b)} \sin^l r \cos^{\frac{d-1}{2}} r \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \\ &\cdot \left(2\psi(n+1) - \psi(a+n) - \psi(b+n) - \log(\cos^2 r)\right) \cos^{2n} r, \end{aligned} \tag{51}$$

where

$$a, b = \frac{1}{2}\left(l + \frac{d-1}{2} \pm \omega\right).$$

One can impose the same boundary conditions as in the non-degenerate case by continuity[15] in E_0. With the frequency given by (43) or (47), (51) gives

$$F_k^{E_0=\frac{d-1}{2}}(r,\theta) = a^{\frac{2-d}{2}} e^{-i\omega_k t} \sin^l r \cos^{\frac{d-1}{2}} r \; P_k^{(l+\frac{d-3}{2},0)}(\cos 2r)\, \mathbf{Y}(\theta). \tag{52}$$

where $\omega_k = \frac{d-1}{2} + l + 2k$. Therefore, at $E_0 = (d-1)/2$ two sets of modes (41) –(45) merge into one, normalizable with respect to the metric (34). Logarithms do not appear in (52). The argument can be repeated for any $m \in Z$ using formulas (110) - (111). We conclude that $E_0 = \frac{d-1}{2} + m, m \in Z$ is *not* a special case and it gives the same Dirichlet or Neumann sets of solutions which were discussed in the previous subsections.

2.4 Singleton modes

From our analysis of the differential equation (26) it follows that at $E_0 = (d-3)/2$ the only set of square-integrable modes is the Dirichlet set (40) – (43) (equivalent to the Neumann set (44) – (47) with $E_0 = (d+1)/2$ corresponding to $so(d-1,2)$ UIR $D(\frac{d+1}{2}, 0)$). We observe therefore that the singleton UIR $D(\frac{d-3}{2}, 0)$ is not realized on $\mathcal{L}^2(CAdS_d)$.

It is known, however, that at $E_0 = (d-3)/2$ the representation becomes indecomposable $D(\frac{d-3}{2}, 0) \rightarrow D(\frac{d+1}{2}, 0)$ ($\frac{d-3}{2}$ "leaking" into $\frac{d+1}{2}$). For example, acting on the ground state $|\frac{d-3}{2}, 0, 0>$ (satisfying $M_i^-|\frac{d-3}{2}, 0, 0>= 0$) the energy-raising operator M_i^+ gives the state $|\frac{d-3}{2}, 1, 0>$ but the action of M_i^- on $|\frac{d-3}{2}, 1, 0>$ does not lead back to $|\frac{d-3}{2}, 0, 0>$ being instead proportional to $|\frac{d+1}{2}, 0, 0>$, the ground state of $D(\frac{d+1}{2}, 0)$. We have therefore a vector space

[15] Alternatively, we may demand the absence of the logarithmic singularity in the solution and set $b = -k$, $k = 0, 1, \ldots$ which gives the same result.

V of states with $E_0 = (d-3)/2, k=0$ (singleton or "physical" modes) and $E_0 = (d+1)/2$ ("gauge" modes) in which states with $E_0 = (d-3)/2, k=0$ do not form an invariant subspace. The Hilbert space for the singleton UIR is a quotient $\mathcal{H} = V/V_{gauge}$.

It is possible to realize this description in terms of the fields in the bulk of $CAdS_d$ satisfying equation (26). In addition to the Dirichlet set (40) – (43) with $E_0 = (d-3)/2$ consider the Neumann set (44) – (47) with $E_0 \to (d-3)/2$. The limit is

$$F^N_{\frac{d-3}{2},l,0}(r,t,\theta) = 0, \tag{53}$$

$$F^N_{\frac{d-3}{2},l,k}(r,t,\theta) = \sqrt{\frac{a^{d-2}k}{\frac{d-3}{2}+l+k}}\, e^{-i\omega_k t}\sin^l r\cos^{\frac{d-3}{2}} r\ P_k^{(l+\frac{d-3}{2},-1)}(\cos 2r)\,\mathbf{Y}(\theta), \tag{54}$$

where $\omega_k = \frac{d-3}{2} + l + 2k$, $k = 1, 2, \ldots$. Using the identity

$$P^{(\alpha,-1)}_{k+1}(\cos 2r) \;=\; \left(\frac{\alpha}{k+1}+1\right)\cos^2 r P_k^{(\alpha,1)}(\cos 2r) \tag{55}$$

one can see that the rest of the $E_0 = (d-3)/2$ Neumann modes (with $k \geq 1$) are equivalent to the standard Dirichlet set (40) – (43) with $E_0 = (d-3)/2$, $k \geq 0$ (or the standard Neumann set (44) – (47) with $E_0 = (d+1)/2$, $k \geq 0$),

$$F^{gauge}_{l,k}(r,t,\theta) \;=\; a^{\frac{d-2}{2}}\sqrt{\frac{\frac{d-1}{2}+l+k}{k+1}}\, e^{-i(\frac{d+1}{2}+l+2k)t}\sin^l r\cos^{\frac{d+1}{2}} r P_k^{(l+\frac{d-3}{2},1)}(\cos 2r)\mathbf{Y}(\theta),\quad k\geq 0. \tag{56}$$

Thus taking the limit shows that for $E_0 = (d-3)/2$ the only set satisfying the equation (26) and Breitenlohner - Freedman boundary conditions is the Dirichlet set.

Relaxing boundary conditions we discover the singleton modes. Indeed, the solution of (26) at $E_0 = (d-3)/2$ is given by

$$\begin{aligned} f(r) \;&=\; \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}\sin^l r\cos^{\frac{d-3}{2}} r \\ &+\; \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)}\sin^l r\cos^{\frac{d+1}{2}} r\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{n!(n+1)!} \\ &\cdot\; \Big(\log(\cos^2 r) - \psi(n+2) - \psi(n+1) + \psi(a+n+1) + \psi(b+n+1)\Big)\cos^{2n} r, \end{aligned} \tag{57}$$

where

$$a, b \;=\; \frac{1}{2}\left(l + \frac{d-3}{2} \pm \omega\right).$$

Instead of imposing vanishing flux boundary conditions (which would require the first term in (57) to vanish and lead to (56)) we require[16] that ω_k be equal to $\omega_k \;=\; \frac{d-3}{2} + l + 2k$. This gives

$$F^{singleton}_l(r,t,\theta) \;=\; e^{-i(\frac{d-3}{2}+l)t}\sin^l r\cos^{\frac{d-3}{2}} r\mathbf{Y}(\theta) \tag{58}$$

[16]Alternatively, we may demand the absence of logarithmic singularity in the solution which gives the same result.

for $k = 0$ and the set (56) for $k > 0$. The singleton modes *are not square-integrable*[17] with respect to (34). Singleton modes (58) and gauge modes (56) form a vector space V which can be equipped with the scalar product

$$(F_k, F_{k'}) = i \lim_{E_0 \to \frac{d-3}{2}} \left(E_0 - \frac{d-3}{2} \right) \int_0^{\pi/2} \left(\frac{\tan r}{a} \right)^{d-2} dr \int d\Omega_{d-2} \left(\bar{F}_k \partial_t F_{k'} - F_{k'} \partial_t \bar{F}_k \right) \tag{59}$$

We observe that $(F^{gauge}_{l,k}, F^{gauge}_{l,k}) = 0$ for all $k \geq 0$ and $(F^{singleton}_l, F^{singleton}_l) = a^{d-2}\left(l + \frac{d-3}{2}\right)$. The singleton modes belong to the Hilbert space $\mathcal{H} = V/V_{gauge}$ with the scalar product (59). Singleton modes have a more singular behavior ($f \sim \cos^{\frac{d-3}{2}} r$) as $r \to \pi/2$ than the gauge modes ($f \sim \cos^{\frac{d+1}{2}} r$) and so at the boundary gauge modes "decouple". In this sense singletons "live on the boundary" of $CAdS_d$.

2.5 Two-point function

In this subsection we shall determine the two-point function,

$$D^{D,N}(x, x') = < 0|\hat{\Phi}^{D,N}(x)\hat{\Phi}^{D,N}(x')|0 >, \tag{60}$$

where

$$\hat{\Phi}^{D,N}(x) = \sum_i \left\{ \phi_i^{D,N} a_i + \bar{\phi}_i^{D,N} a_i^* \right\}, \tag{61}$$

and $\left\{ \phi_i^{D,N} \right\}$ is a complete set (Dirichlet or Neumann) of the solutions of (26) on $CAdS_d$. We shall do this calculation twice - first by generalizing approach of [30] to d dimensions and then by evaluating the sum

$$D^{D,N}(x, x') = \sum_i \phi_i^{D,N}(x)\bar{\phi}_i^{D,N}(x') \tag{62}$$

explicitly. The second method has an advantage of providing explicit normalization for $D(x, x')$ determined by the boundary conditions imposed on the fields $\phi_i(x)$.

Due to the $SO(d-1, 2)$ invariance, $D(x, x')$ is a function of $Z = a^2 X \cdot X'$ [30]. In global coordinates (7) - (9) Z reads

$$Z = \sec r \sec r' \left(\cos(t - t') - \frac{\sin r \sin r'}{rr'} z_i z_i' \right). \tag{63}$$

The two-point function $D(Z)$ satisfies (26),

$$\left[\left(1 - Z^2\right) \partial^2_{ZZ} - dZ\partial_Z + E_0 \left(E_0 - d + 1\right) \right] D(Z) = 0, \tag{64}$$

[17]Note that the singleton modes do not satisfy vanishing flux boundary conditions on $CAdS_d$.

general solution of which can be written as

$$D(Z) \;=\; C_1(E_0,d)\;\; {}_2F_1\left(\frac{d-1-E_0}{2},\frac{E_0}{2};\frac{1}{2};Z^2\right) \;+\; C_2(E_0,d)\;\; Z\;\; {}_2F_1\left(\frac{d-E_0}{2},\frac{E_0+1}{2};\frac{3}{2};Z^2\right). \tag{65}$$

Two independent solutions correspond to Dirichlet and Neumann sets of modes. To make connection with [30] it is convenient to rewrite (64) –(65) in terms of $\xi = 1/Z$:

$$\left[\xi^2(\xi^2-1)\partial^2_{\xi\xi} + \xi\left(2\xi^2+d-2\right)\partial_\xi + E_0\left(E_0-d+1\right)\right] D(\xi) = 0, \tag{66}$$

$$\begin{aligned} D(Z) \;&=\; C_1(E_0,d)\; Z^{E_0+1-d}\;\; {}_2F_1\left(\frac{d-1-E_0}{2},\frac{d-E_0}{2};\frac{d+1}{2}-E_0;\frac{1}{Z^2}\right) \\ &+\; C_2(E_0,d)\; Z^{-E_0}\;\; {}_2F_1\left(\frac{E_0+1}{2},\frac{E_0}{2};\frac{3-d}{2}+E_0;\frac{1}{Z^2}\right). \end{aligned} \tag{67}$$

Now we shall calculate $D^D(Z)$ and $D^N(Z)$ using (61) and explicit solutions found in this section. Putting for simplicity $x' = 0$ we have $Z = \sec r \cos t$. Then

$$D^D(r,t) = \frac{\Gamma\left(\frac{d-1}{2}+1\right)}{(d-1)\pi^{\frac{d-1}{2}}} \cos^{d-1-E_0} r\, e^{-it(d-1-E_0)} \sum_{k=0}^{\infty} C^2_{E_0,0,k} \frac{\left(\frac{d-1}{2}\right)_k}{k!} P_k^{(l+\frac{d-3}{2},\frac{d-1}{2}-E_0)}(\cos 2r) e^{-i2kt}. \tag{68}$$

Performing the sum with the help of [34] we get

$$D^D(Z) = \frac{a^{d-2}}{2^{d-E_0}\pi^{\frac{d-1}{2}}} \frac{\Gamma\left(d-1-E_0\right)}{\Gamma\left(\frac{d+1}{2}-E_0\right)} Z^{E_0-d+1} {}_2F_1\left(\frac{d-1-E_0}{2},\frac{d-E_0}{2};\frac{d+1}{2}-E_0;\frac{1}{Z^2}\right). \tag{69}$$

For the Neumann set we have

$$D^N(Z) = \frac{a^{d-2}}{2^{E_0+1}\pi^{\frac{d-1}{2}}} \frac{\Gamma\left(E_0\right)}{\Gamma\left(E_0-\frac{d-3}{2}\right)} Z^{-E_0} {}_2F_1\left(\frac{E_0}{2},\frac{E_0+1}{2};E_0-\frac{d-3}{2};\frac{1}{Z^2}\right). \tag{70}$$

Explicit expressions for C_1 and C_2 allow us to consider the limit $E_0 \to (d-3)/2$:

$$\lim_{E_0\to\frac{d-3}{2}} D^D(Z) \;=\; D^D_{\frac{d-3}{2}}(Z) \;=\; \frac{a^{d-2}\Gamma(\frac{d+1}{2})}{(2\pi)^{4\frac{d-1}{2}}}\; Z^{-\frac{d+1}{2}} {}_2F_1\left(\frac{d+1}{4},\frac{d+3}{4};2;\frac{1}{Z^2}\right), \tag{71}$$

$$\lim_{E_0\to\frac{d-3}{2}} D^N(Z) \;=\; D^D_{\frac{d-3}{2}}(Z). \tag{72}$$

Thus the limit of the Neumann two-point function is simply equal to the Dirichlet two-point function at $E_0 = (d-3)/2$. This result is obviously consistent with the description of the limit (54) of the Neumann set satisfying vanishing flux boundary conditions.

Relaxing the boundary conditions and solving (66) *directly at* $E_0 = (d-3)/2$ we obtain

$$D(Z) \;=\; C_1 w_1(Z) \;+\; C_2 w_2(Z), \tag{73}$$

where

$$w_1 \;=\; Z^{-\frac{d+1}{2}}\, {}_2F_1\left(\frac{d+1}{4}, \frac{d+3}{4}; 2; \frac{1}{Z^2}\right), \tag{74}$$

$$\begin{aligned} w_2 \;&=\; -Z^{-\frac{d+1}{2}} \log Z^2\, {}_2F_1\left(\frac{d+1}{4}, \frac{d+3}{4}; 2; \frac{1}{Z^2}\right) + Z^{-\frac{d+1}{2}} \sum_{n=1}^{\infty} Z^{-2n}\, \frac{\left(\frac{d+1}{4}\right)_n \left(\frac{d+3}{4}\right)_n}{(2)_n n!} \\ &\left[\psi\left(\frac{d+1}{4}\right) + \psi\left(\frac{d+3}{4}\right) - \psi\left(\frac{d+1}{4}\right) - \psi\left(\frac{d+3}{4}\right) + \psi(2) - \psi(2+n) + \psi(1) - \psi(n+1)\right] \\ &+\; Z^{-\frac{d-3}{2}}\, \frac{16}{(d-1)(d-3)}. \end{aligned} \tag{75}$$

with coefficients C_1 and C_2 remaining undetermined. The solution (71) is recovered by putting $C_2 = 0$ which is equivalent to imposing vanishing flux boundary conditions. Another possibility is to relax vanishing flux boundary conditions and introduce the singleton modes (58) propagating in the bulk [30]. A fourth-order differential equation was proposed in [30] to describe the singleton field theory in the bulk of $CAdS_4$. The original motivation in [30] was to find an equation for the singleton two-point function solution of which would be free from logarithmic singularities. Even though this goal cannot be achieved globally[18] on $CAdS_d$, the proposed equation provides an interesting opportunity for quantization of the singleton modes in the bulk via Gupta - Bleuler triplet technique [30],[33],[4]. We shall obtain explicit solutions of this equation in the next section.

3 Flato — Fronsdal wave equation

A fourth-order wave equation for singleton modes,

$$\left(\Box_{CAdS_d} - \lambda\right)^2 F \;=\; 0, \tag{76}$$

was proposed by Flato and Fronsdal [30] and used in a number of publications. Here we obtain solutions of this equation and discuss their properties.

We are looking for solutions of (76) of the form $F = f(r)e^{-i\omega t}\mathbf{Y}(\theta)$. Equation for $f(r)$ reads

$$Q^2 f \;=\; a_4(r) f^{IV} \;+\; a_3(r) f'''(r) \;+\; a_2(r) f''(r) \;+\; a_1(r) f'(r) \;+\; a_0(r) f(r) \;=\; 0, \tag{77}$$

where

$$a_4 \;=\; \cos^4 r, \tag{78}$$

$$a_3 \;=\; 2(d-2)\cos^2 r \cot r - 4\cos^3 r \sin r \tag{79}$$

[18] The two-point function ${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{Z^2}\right)$, obtained in [30] as the solution of the fourth-order equation avoids logarithmic singularity similar to that one in (75) at the *boundary* of $CAdS_4$ but instead it acquires logarithmic behavior near the origin.

$$
\begin{aligned}
a_2 \;&=\; -2(d-2)\cos^2 r - 2\frac{\lambda}{a^2}\cos^2 r - 2\cos^4 r + 2\omega^2\cos^4 r - 2(d-2)\cot^2 r \\
&+\; (d-2)^2\cot^2 r + 2\Lambda\cos^2 r\cot^2 r + 2\cos^2 r\sin^2 r
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
a_1 \;&=\; -2(d-2)\lambda\cot r + 2(d-2)\omega^2\cos^2 r\cot r + 2(d-2)\cot^3 r - 4\Lambda\cot^3 r \\
&+\; 2(d-2)\Lambda\cot^3 r - (d-2)^2\cot r\csc^2 r - 4\omega^2\cos^3 r\sin r
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
a_0 \;&=\; 2\Lambda\cot^2 r\csc^2 r - 2(d-2)\omega^2\cos^2 r - 2\frac{\lambda}{a^2}\omega^2\cos^2 r - 2\omega^2\cos^4 r - 2\Lambda\frac{\lambda}{a^2}\cot^2 r \\
&+\; 2\Lambda\omega^2\cos^2 r\cot^2 r + \omega^4\cos^4 r + 4\Lambda\cot^4 r + \Lambda^2\cot^4 r - 2(d-2)\Lambda\cot^2 r\csc^2 r \\
&+\; 2\omega^2\cos^2 r\sin^2 r + \frac{\lambda^2}{a^4}
\end{aligned} \tag{82}
$$

where Λ is the eigenvalue of $\triangle_{S^{d-2}}$ (27). Roots of the indicial equation at $r=0$ are:

$$
\alpha_1^0 \;=\; l, \quad \alpha_2^0 \;=\; l+2, \quad \alpha_3^0 \;=\; 3-d-l, \quad \alpha_4^0 \;=\; 5-d-l. \tag{83}
$$

Roots of the indicial equation at $r=\pi/2$ are:

$$
\alpha_{1,2}^{\pi/2} = \frac{d-1}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 + \frac{\lambda}{a^2}}. \tag{84}
$$

$$
\alpha_{3,4}^{\pi/2} \;=\; \alpha_{1,2}^{\pi/2} \tag{85}
$$

The generic solution with nonsingular behavior at the origin can be written as

$$
f(r) \;=\; A\psi_1(r) \;+\; B\psi_2(r), \tag{86}
$$

where ψ_1 is given by (36) and ψ_2 satisfies $Q\psi_2 = c\psi_1$, where c is an arbitrary constant. Solution nonsingular at $r=0$ can easily be constructed[19]

$$
\psi_2 \;=\; \sin^{l+2} r\cos^{E_0} r\left(1 \;+\; c_1\sin^2 r \;+\; c_2\sin^4 r \;+\; \cdots\right), \tag{87}
$$

where

$$
c_1 \;=\; \frac{(E_0+l+1)^2 + 2l + d - \omega^2}{2(2l+d+1)}, \tag{88}
$$

$$
c_2 \;=\; \frac{\left(\omega^2 - (E_0+l+2)^2 - 2l - d - 3\right)^2 + r_0}{8(2l+d+1)(2l+d+3)}, \tag{89}
$$

$$
r_0 = -\frac{1}{3}\left(16E_0^2 + 8E_0(6l+d+11) + 12l^2 - 12l(d-9) - 5(d-1)^2 + 128\right). \tag{90}
$$

[19]The solution cannot be written in terms of hypergeometric function for the generic value of E_0.

When $E_0 = (d-3)/2$ the above solution reduces to

$$\psi_2 \;=\; \sin^{l+2} r \cos^{\frac{d-3}{2}} r \;{}_2F_1\left(\frac{l}{2} + \frac{d+1}{4} + \frac{\omega}{2}, \frac{l}{2} + \frac{d+1}{4} - \frac{\omega}{2}; l + \frac{d+1}{2}; \sin^2 r\right). \tag{91}$$

With the "singleton frequency", $\omega_k = (d-3)/2 + l + 2k$, ψ_2 gives two sets of modes,

$$F^{scalar}_{l,k} \;=\; \psi_2^{(k)} e^{-i(\frac{d+1}{2}+l+2k)t}\mathbf{Y}(\theta) = e^{-i(\frac{d+1}{2}+l+2k)t} \sin^{l+2} r \cos^{\frac{d-3}{2}} r \; P_k^{(l+\frac{d-1}{2},0)}(\cos 2r)\,\mathbf{Y}(\theta) \tag{92}$$

and

$$\Psi^{(0)}_l = e^{-i(\frac{d-3}{2}+l)t} \sin^{l+2} r \cos^{\frac{d-3}{2}} r \; \Phi\left(\sin^2 r, 1, l + \frac{d-1}{2}\right)\mathbf{Y}(\theta), \tag{93}$$

where $k = 0, 1, \ldots$, $\Phi(z, s, a) = \sum_{n=0}^{\infty} z^n/(n+a)^s$. At the same time solution ψ_1 produces singleton modes $F_l^{singleton}$ (58) and gauge modes $F_{k,l}^{gauge}$ (56). The scalar, singleton and gauge modes form the Gupta - Bleuler triplet,

$$\left(\frac{d+1}{2}, scalar\right) \to \left(\frac{d-3}{2}, singleton\right) \to \left(\frac{d+1}{2}, gauge\right). \tag{94}$$

which can be verified explicitly by acting on the states $|E_0, l, k>$ with the generators $M_i^{\pm}$[20].

Let us examine the set $\{\Psi_l^{(0)}\}$ more closely. Explicit expressions in odd/even dimension are:

$$\Psi^{(0)}_l = -\sin^{l+2} r \cos^{n-1} r \frac{l+n}{\sin^{2(l+n)} r}\left[\log\left(\cos^2 r\right) + \sum_{k=1}^{l+n-1} \frac{\sin^{2k} r}{k}\right], \tag{95}$$

for $d = 3, 5, \ldots 2n+1$,

$$\Psi^{(0)}_l = \sin^{l+2} r \cos^{n-3/2} r \frac{2(l+n)-1}{\sin^{2(l+n)-1} r}\left[\operatorname{arctanh}(\sin r) + \sum_{k=1}^{l+n-1} \frac{\sin^{2k-1} r}{2k-1}\right]. \tag{96}$$

for $d = 2, 4, \ldots 2n$. In four dimensions, the two lowest states are given by

$$\Psi^{(0)}_0 = 3e^{-it/2} \cos^{1/2} r \left[\frac{\operatorname{arctanh}(\sin r)}{\sin r} - 1\right], \tag{97}$$

$$\Psi^{(0)}_1 = 5e^{-i3t/2} \cos^{1/2} r \left[\frac{\operatorname{arctanh}(\sin r)}{\sin^2 r} - \frac{1}{\sin r} - \frac{1}{3}\sin r\right]\sin\theta. \tag{98}$$

For $r < \pi/2$ they can be written in the form of power series

$$\Psi^{(0)}_0 = e^{-it/2} \sin^2 r \cos^{1/2} r \left(1 + \frac{3}{5}\sin^2 r + \frac{3}{7}\sin^4 r + \cdots\right), \tag{99}$$

$$\Psi^{(0)}_1 = e^{-i3t/2} \sin^3 r \cos^{1/2} r \left(1 + \frac{5}{7}\sin^2 r + \frac{5}{9}\sin^4 r + \cdots\right)\sin\theta. \tag{100}$$

[20] For example, in $d = 4$ one has $M_3^-|\frac{5}{2}, 0, 0; \mathrm{scal}> = M_3^-(\sin^2 r e^{-i5t/2}\cos^{1/2} r) = 2ie^{-i3t/2}\sin r \cos^{1/2} r \cos\theta$ $= 2i|\frac{1}{2}, 1, 0, \mathrm{singleton}>$ and $M_3^+|\frac{5}{2}, 0, 0; \mathrm{scal}> = -i/2e^{-i7t/2}\cos^{1/2} r \cos\theta \sin r(5\cos 2r - 1) = |\frac{5}{2}, 1, 0; \mathrm{gauge}>$ $\oplus|\frac{5}{2}, 1, 0; \mathrm{scal}>$.

While the energy raising operators act on (97), (98) by the rule $M_i^+|\Psi_l^{(0)} >= |\Psi_l^{(0)} > \oplus|\text{singleton} >$[21] the action of the energy lowering operators gives the states with negative energy[22]. The set $\left\{\Psi_l^{(0)}\right\}$ is therefore unacceptable and should be eliminated. This can be done by choosing

$$\psi_2 \;=\; \frac{\sin^{l+2} r \cos^{\frac{d-3}{2}} r}{\Gamma\left(\omega - \frac{d-3}{2} - l\right)}\; {}_2F_1\left(\frac{l}{2} + \frac{d+1}{4} + \frac{\omega}{2}, \frac{l}{2} + \frac{d+1}{4} - \frac{\omega}{2}; l + \frac{d+1}{2}; \sin^2 r\right). \tag{101}$$

rather than (91) as the second fundamental nonsingular at the origin solution of (77).

Finally, the space of solutions of the fourth-order equation (77) is given by

$$W = \left\{F^{scalar}\right\} \oplus \left\{F^{singleton}\right\} \oplus \left\{F^{gauge}\right\}. \tag{102}$$

with $F_{l,k}^{scalar}$, $F_l^{singleton}$ and $F_{l,k}^{gauge}$ given correspondingly by (92), (58) and (56).

Note that if the vanishing flux boundary conditions are imposed, the only solution of the fourth-order equation is the Dirichlet set (56). In that case the solution ψ_2 is trivial and thus the fourth-order equation gives nothing new in comparison with the second-order one.

Gupta - Bleuler triplet and the singleton two-point function According to the description of the singleton representation realized in the bulk of $CAdS$, the singleton two-point function satisfies Flato - Fronsdal wave equation (76). It is straightforward to generalize the four-dimensional result of [30] to arbitrary d,

$$D_{FF}(Z) \;=\; Z^{-\frac{d-3}{2}}\; {}_2F_1\left(\frac{d-3}{4}, \frac{d-1}{4}; 1; \frac{1}{Z^2}\right). \tag{103}$$

Having obtained explicit expressions for the modes we can now demonstrate that (103) admits decomposition of the form (62) with $\{\phi_i\}$ being the members of the Gupta - Bleuler triplet (102). Indeed,

$$D_{FF}(Z) = 2^{\frac{d-3}{2}} \cos^{\frac{d-3}{2}} r e^{-i\frac{d-3}{2}t} \sum_{n=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_n}{n!} e^{-i2nt} P_n^{(\frac{d-5}{2},0)}(\cos 2r). \tag{104}$$

Using (55) it is not difficult to obtain the identity

$$P_{k+1}^{(\alpha-1,0)}(\cos 2r) \;=\; \frac{\alpha}{k+1} \cos^2 r P_k^{(\alpha,1)}(\cos 2r) - \sin^2 r P_k^{(\alpha+1,0)}(\cos 2r). \tag{105}$$

Then

$$\begin{aligned} D_{FF}(Z) \;&=\; 2^{\frac{d-3}{2}} \cos^{\frac{d-3}{2}} r e^{-i\frac{d-3}{2}t} \\ &+\; 2^{\frac{d-3}{2}} \cos^{\frac{d+1}{2}} r e^{-i\frac{d+1}{2}t} \left(\frac{d-3}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{n+1}}{n!(n+1)^2} e^{-i2nt} P_n^{(\frac{d-3}{2},1)}(\cos 2r) \\ &-\; 2^{\frac{d-3}{2}} \cos^{\frac{d-3}{2}} r \sin^2 r e^{-i\frac{d+1}{2}t} \sum_{n=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{n+1}}{n!(n+1)^2} e^{-i2nt} P_n^{(\frac{d-1}{2},0)}(\cos 2r). \end{aligned} \tag{106}$$

[21] For example, $M_3^+\Psi_0^{(0)} = 3i/5\Psi_1^{(0)} - 2iF_1^{singleton}$.

[22] $M_3^-\Psi_0^{(0)} \sim e^{it/2}$

Three terms of this expression clearly correspond to the contribution of the singleton (58), gauge (56) and scalar (92) modes.

A Transformation properties of ${}_2F_1(a,b;c;z)$

In this appendix we record some useful formulae concerning the well-known transformation properties of the hypergeometric function.

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z) \tag{107} $$

$$ \begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a,b;a+b-c+1;1-z) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, {}_2F_1(c-a,c-b;c-a-b+1;1-z) \end{aligned} \tag{108} $$

$$ \begin{aligned} {}_2F_1(a,b,a+b;z) &= \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2} \\ &\cdot\ (2\psi(n+1)-\psi(a+n)-\psi(b+n)-\log(1-z))\,(1-z)^n, \end{aligned} \tag{109} $$

$$ \begin{aligned} {}_2F_1(a,b,a+b+m;z) &= \frac{\Gamma(m)\,\Gamma(a+b+m)}{\Gamma(a+m)\,\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{n!(1-m)_n}(1-z)^n \\ &- \frac{\Gamma(a+b+m)}{\Gamma(a)\,\Gamma(b)}(z-1)^m\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{n!(m+n)!} \\ &\cdot\ [-\psi(n+1)-\psi(n+m+1)+\psi(a+n+m)+\psi(b+n+m) \\ &+ \log(1-z)](1-z)^n, \end{aligned} \tag{110} $$

$$ \begin{aligned} {}_2F_1(a,b,a+b-m;z) &= \frac{\Gamma(m)\,\Gamma(a+b-m)}{\Gamma(a)\,\Gamma(b)}(1-z)^{-m}\sum_{n=0}^{m-1}\frac{(a-m)_n(b-m)_n}{n!(1-m)_n}(1-z)^n \\ &- (-1)^m\frac{\Gamma(a+b-m)}{\Gamma(a-m)\,\Gamma(b-m)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(m+n)!}[-\psi(n+1) \\ &- \psi(n+m+1)+\psi(a+n+m)+\psi(b+n+m)+\log(1-z)] \\ &\cdot\ (1-z)^n, \end{aligned} \tag{111} $$

B Two-dimensional case

Two-dimensional case is slightly different because in $d=2$ both solutions of the equation (28) are nonsingular at the origin.

The eigenvalue equation (26) reads

$$\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial r^2} + \frac{\lambda}{a^2 \cos^2 r} F = 0, \tag{112}$$

where $\lambda = a^2 E_0(E_0 - 1)$. The solution is given by $F(r,t) = e^{-i\omega t} f(r)$, where

$$f(r) = A f_1(r) + B f_2(r), \tag{113}$$

$$f_1(r) = \cos^{E_0} r\, {}_2F_1\left(\frac{E_0 - \omega}{2}, \frac{E_0 + \omega}{2}; \frac{1}{2}; \sin^2 r\right), \tag{114}$$

$$f_2(r) = \sin r \cos^{1-E_0} r\, {}_2F_1\left(\frac{E_0 + 1 - \omega}{2}, \frac{E_0 + 1 + \omega}{2}; \frac{3}{2}; \sin^2 r\right). \tag{115}$$

Since the origin is a singular point of the limit-circle type we have to specify boundary conditions. Dirichlet condition $f(0) = 0$ singles out solution (115). Then the vanishing flux boundary conditions at $r = \pi/2$ give frequency quantization $\omega_k = 2 - E_0 + 2k$, $k = 0, 1, \ldots$ and eigenfunctions

$$f_k^{DD}(r) = C_k \sin r \cos^{1-E_0} r P_k^{(\frac{1}{2}, \frac{1}{2} - E_0)}(\cos 2r), \tag{116}$$

where $C_k^2 = \Gamma(k - E_0 + 2)\, k! / \Gamma(k + 3/2)\, \Gamma(k - E_0 + 1/2)$ (Dirichlet-Dirichlet set) or $\omega_k = E_0 + 2k + 1$,

$$f_k^{DN}(r) = C_k \sin r \cos^{E_0} r P_k^{(\frac{1}{2}, E_0 - \frac{1}{2})}(\cos 2r), \tag{117}$$

where $C_k^2 = \Gamma(k + E_0 +!)\, k! / \Gamma(k + 3/2)\, \Gamma(k + E_0 + 1/2)$ (Dirichlet-Neumann set), $k = 0, 1, \ldots$. Neumann condition $f'(0) = 0$ singles out (114). We obtain then $\omega_k = 1 - E_0 + 2k$,

$$f_k^{ND}(r) = C_k \cos^{1-E_0} r P_k^{(-\frac{1}{2}, \frac{1}{2} - E_0)}(\cos 2r), \tag{118}$$

where $C_k^2 = \Gamma(k - E_0 + 1)\, k! / \Gamma(k + 1/2)\, \Gamma(k - E_0 + 3/2)$ (Neumann-Dirichlet set) or $\omega_k = E_0 + 2k$,

$$f_k^{NN}(r) = C_k \cos^{E_0} r P_k^{(-\frac{1}{2}, E_0 - \frac{1}{2})}(\cos 2r), \tag{119}$$

where $C_k^2 = \Gamma(k + E_0)\, k! / \Gamma(k + 1/2)\, \Gamma(k + E_0 + 1/2)$ (Neumann-Neumann set), $k = 0, 1, \ldots$. These solutions correspond to the discrete series of UIRs of $\widetilde{SO}(1,2)$. There are no positive energy singleton modes in this case since $E_0^{min} = -1/2 < 0$.

C Radial part of the Laplace equation on $CAdS_d$

Unusual properties of the singleton modes can be further emphasized by considering the following example. One can redefine function $f(r)$ to absorb the measure in (34), $f(r) = \tan^{1-d/2} r \phi(r)$. The square-integrability condition for ϕ is $\int_0^{\pi/2} \phi^2 dr < \infty$ and equation (28) reduces to the Schrödinger-type one

$$\left(-\frac{d^2}{dr^2} + V(r)\right)\phi = E,$$

where $E = \omega^2$,

$$V(r) \;=\; \frac{l(l+d-3)}{\sin^2 r} + \frac{(d-2)(d-4)}{\sin^2 2r} - \frac{2-d-E_0(E_0-d+1)}{\cos^2 r}.$$

One finds that the "energy" E is quantized, $E_l^{(k)} = (E_0 + l + 2k)^2$, $k = 0, 1, \ldots$.

For the singleton representation $E_0 = (d-3)/2$. The set of formal solutions in this case consists of "singleton modes" ($k = 0$) and "gauge modes" ($k > 1$). Simple analysis shows that "singleton modes" correspond to the "energy" levels $E_l^{(0)}$ which lie entirely below the minimal level of the potential (see Figure (2)). These states cannot be given neither classical nor quantum-mechanical interpretation.

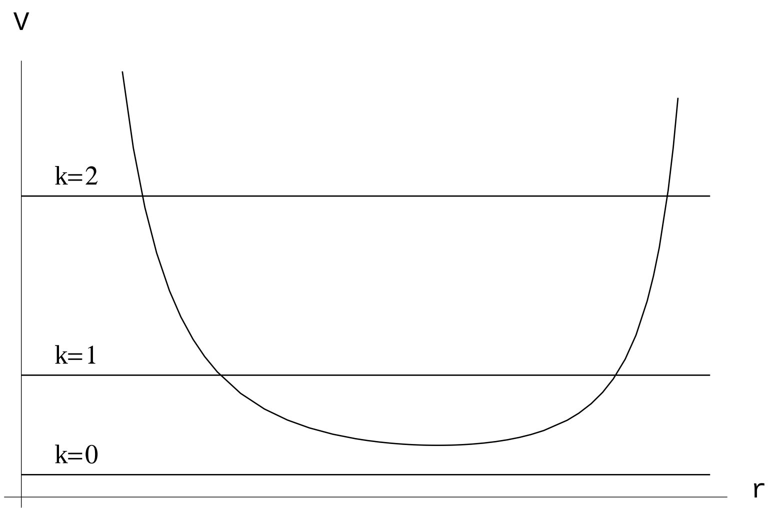

Figure 2: Effective potential and "energy" levels for the singleton representation with $E_0 = (d-3)/2$. Singleton wave function would have corresponded to the $k = 0$ level.

Acknowledgments

I would like to thank Massimo Porrati, Martin Schaden and especially Christian Fronsdal for useful and stimulating discussions, and Daniel Sternheimer for correspondence.

References

[1] J. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) hep-th/9711200; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from noncritical

string theory," Phys. Lett. **B428**, 105 (1998) hep-th/9802109; E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) hep-th/9802150.

[2] V. Balasubramanian, P. Kraus and A. Lawrence, "Bulk vs. boundary dynamics in anti-de Sitter space-time," Phys. Rev. **D59**, 046003 (1999) hep-th/9805171.

[3] I.R.Klebanov and E.Witten, "AdS/CFT correspondence and symmetry breaking", hep-th/9905104.

[4] M. Flato, C. Fronsdal and D. Sternheimer, "Singleton physics," hep-th/9901043.

[5] I.I. Kogan, "Singletons and logarithmic CFT in AdS / CFT correspondence," Phys. Lett. **B458**, 66 (1999) hep-th/9903162.

[6] S.W. Hawking and G.F.R. Ellis, "The Large Scale Structure of Space-time", Cambridge University Press (1973).

[7] S.J. Avis, C.J. Isham, and D. Storey, "Quantum field theory in anti-de Sitter spacetime", Phys.Rev. **D18**, 3565 (1978).

[8] P. Breitenlohner and D.Z. Freedman, "Stability in gauged extended supergravity", Ann.Phys.**144**, 197 (1982).

[9] N. Vilenkin and A. Klimuk, "Representations of Lie Groups and Special Functions", Kluwer Academic Publishers, 1993; A. Barut and R. Raczka, "Theory of Group Representations and Applications", Polish Scientific Publishers, 1977.

[10] N.T. Evans, "Discrete series for the universal covering group of the $3+2$ de Sitter group", J.Math.Phys. **8**, 170 (1967).

[11] S. Minwalla, "Restrictions imposed by superconformal invariance on quantum field theories," Adv. Theor. Math. Phys. **2**, 781 (1998) hep-th/9712074.

[12] J.B. Ehrman, "On the unitary irreducible representations of the universal covering group of the $3+2$ de Sitter group", Proc. Cambridge Philos. Soc., **53**, 290 (1957).

[13] P.A.M. Dirac, "A remarkable representation of the 3+2 de Sitter group", J.Math.Phys., **4**, 901 (1963).

[14] M. Günaydin and N.P. Warner, "Unitary Supermultiplets Of Osp(8/4,R) And The Spectrum Of The S(7) Compactification Of Eleven-Dimensional Supergravity," Nucl. Phys. **B272**, 99 (1986); M. Günaydin, P. van Nieuwenhuizen and

N.P. Warner, "General Construction Of The Unitary Representations Of Anti-De Sitter Superalgebras And The Spectrum Of The S**4 Compactification Of Eleven-Dimensional Supergravity," Nucl. Phys. **B255**, 63 (1985).

[15] M. Flato and C. Fronsdal, "One massless particle equals two Dirac singletons", Lett.Math.Phys., 2 (1978) 421.

[16] H. Nicolai, "Representations of supersymmetry in anti-de Sitter space", in "Supersymmetry and Supergravity '84", Eds.B. de Wit, P.Fayet, P. van Nieuwenhuizen, World Scientific, 1984.

[17] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, "Supersymmetric Supermembrane Vacua And Singletons," Phys. Lett. **199B**, 69 (1987); M.P. Blencowe and M.J. Duff, "Supersingletons," Phys. Lett. **203B**, 229 (1988). E. Bergshoeff, A. Salam, E. Sezgin and Y. Tanii, "Singletons, Higher Spin Massless States And The Supermembrane," Phys. Lett. **205B**, 237 (1988).

[18] S. Ferrara, C. Fronsdal and A. Zaffaroni, "On N=8 supergravity on AdS(5) and N=4 superconformal Yang-Mills theory," Nucl. Phys. **B532**, 153 (1998) hep-th/9802203.

[19] S. Ferrara and A. Zaffaroni, "Bulk gauge fields in AdS supergravity and supersingletons," hep-th/9807090.

[20] C. Fronsdal, "Elementary Particles in Curved space.4.Massless Particles", Phys.Rev. **D12**, 3819 (1975)

[21] L. Mezinescu and P.K. Townsend, "Stability at a local maximum in higher-dimensional anti-de Sitter space and applications to supergravity", Ann.Phys. **160**, 406 (1985)

[22] S.W. Hawking, "The boundary conditions for gauged supergravity", Phys.Lett. **126**, 175 (1983)

[23] E. Angelopoulos, M. Flato, C. Fronsdal, D. Sternheimer, "Massless particles, conformal group and De Sitter universe", Phys. Rev.**D23**, 1278 (1981).

[24] E.A.Coddington and N.Levinson, "Theory of ordinary differential equations", McGraw-Hill (1955).

[25] N.D. Birrell and P.C.W. Davies, "Quantum Fields in Curved Space", Cambridge University Press (1982).

[26] V. Bargmann, “Irreducible unitary representations of the Lorentz group”, Ann.Math., **48**, 568 (1947).

[27] M. Flato and C. Fronsdal, “Quantum field theory of singletons. The Rac”, J.Math.Phys. **22**, 1100 (1981).

[28] C. Fronsdal, “Dirac supermultiplet”, Phys.Rev. **D26**, 1988 (1982).

[29] L. Castell and W. Heidenreich, “SO(3,2) Invariant Scattering And Dirac Singletons,” Phys. Rev. **D24**, 371 (1981).

[30] M. Flato and C. Fronsdal, “The singleton dipole”, Comm.Math.Phys. **108**, 469 (1987).

[31] S. Ferrara and C. Fronsdal, “Gauge fields and singletons of AdS(2p+1),” Lett. Math. Phys. **46**, 157 (1998) hep-th/9806072.

[32] M. Flato and C. Fronsdal, “Interacting singletons,” Lett. Math. Phys. **44**, 249 (1998) hep-th/9803013.

[33] H. Araki, “Indecomposable representations with invariant inner product. A theory of the Gupta - Bleuler triplet”, Comm.Math.Phys. **97** 149 (1985).

[34] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, “Integraly i ryady. Specialnye funktsii”, Nauka, Moscow, 1983. English translation: “Integrals and series. Vol.2:Special Functions”, Gordon & Breach Science Publishers, New York, 1988.